Real Analysis II
Problem Set 2: Topology II

Exam style questions are marked with (*).

Trigonometry

Question 1. What is $\arctan(1)$?

Question 2. Find appropriate ranges and then graph the following functions:

- $f(x) = (x+1)\arctan\frac{1}{1+x^2}$;
- $f(x) = \ln\big(\sin(x)\big)$;
- $f(x) = \ln\big(\cos(x)\big)$;
- $f(x) = \lim\limits_{n\to\infty} x\arctan(nx)$;
- $f(x) = e^{\sin(x)}$.

Continuity

Definition. For a real function $f : A \to \mathbb{R}$ and a value $c \in A$ we write

$$f(-c) = \lim_{x\to c} f(x),\ \ x < c,$$
$$f(+c) = \lim_{x\to c} f(x),\ \ x > c,$$

for the left- and right-handed limits of f respectively. We say f has:

- a *discontinuity of the first kind* at c if $f(c)$, $f(-c)$, and $f(+c)$ are real numbers and have different values (i.e. the limits both exist). If $f(-c) = f(+c)$ we say f has a *removable discontinuity* at c, else we say f has a *jump discontinuity* at c;
- a *discontinuity of the second kind* at c if either of the limits $f(-c)$ or $f(+c)$ do not exist.

Question 3. Let $f : A \to \mathbb{R}$ have a removable singularity at $c \in A$. Prove that the augmented function

$$\tilde{f} : A \to \mathbb{R} : x \mapsto \begin{cases} f(x), & x \neq c, \\ f(-c), & x = c, \end{cases}$$

is continuous at c.

Question 4. Decide if the following functions are continuous. If not, classify their discontinuities. If the discontinuities are removable, remove them.

- $f(x) = x - \lfloor x \rfloor$;
- $f(x) = e^{\frac{1}{x+1}}$;
- $f(x) = \operatorname{sgn}(x)$, where
$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0; \end{cases}$$
- $f : [0,\infty) \to \mathbb{R} : x \mapsto |\operatorname{sgn}(x)|$;

- $f(x) = \begin{cases} x^2, & x \leq 3, \\ 2x+1, & x > 3; \end{cases}$;

Question 5. Investigate (and give examples where appropriate) whether two discontinuous functions f and g can form a continuous function of the form:

- $\lambda f(x)$, for any $\lambda \in \mathbb{R}$;
- $f(x) + g(x)$;
- $f(x)g(x)$;
- $\frac{f(x)}{g(x)}$.